

Greedy palindromic lengths^{*†}

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Abstract

In [A. Frid, S. Puzynina, L.Q. Zamboni, *On palindromic factorization of words*, Adv. in Appl. Math. 50 (2013), 737-748], it was conjectured that any infinite word whose palindromic lengths of factors are bounded is ultimately periodic. We introduce variants of this conjecture and prove this conjecture in particular cases. Especially we introduce left and right greedy palindromic lengths. These lengths are always greater than or equals to the initial palindromic length. When the greedy left (or right) palindromic lengths of prefixes of a word are bounded then this word is ultimately periodic.

1 Introduction

A fundamental question in Combinatorics on Words is how words can be decomposed on smallest words. For instance, readers can think to some topics presented in the first Lothaire's book [14] like Lyndon words, critical factorization theorem, equations on words, or to the theory of codes [2, 3], or to many related works published since these surveys. As another example, let us mention that in the area of Text Algorithms some factorizations like Crochemore or Lempel-Ziv factorizations play an important role [6, 12]. These factorizations have been extended to infinite words and, for the Fibonacci word, some links have been discovered with the Wen and Wen's decomposition in singular words [4] (see also [10] for a generalization to Sturmian words and see [17, 13] for more on singular words).

In their article [9], A.E. Frid, S. Puzynina and L.Q. Zamboni defined the *palindromic length* of a finite word w as the least number of (nonempty) palindromes needed to decompose w . More precisely the palindromic length of w is the least number k such that $w = \pi_1 \cdots \pi_k$ with π_1, \dots, π_k palindromes. As in the article [9] we let $|w|_{pal}$ denote the palindromic length of w . For instance $|abaab|_{pal} = 2$. The palindromic length of a finite word can also be defined inductively by $|\varepsilon|_{pal} = 0$ (where ε denotes the empty word), and, for any nonempty word u , $|u|_{pal} = \min\{|p|_{pal} + 1 \mid u = p\pi \text{ and } \pi \text{ is a palindromic nonempty suffix of } u\}$. This recursive definition is the starting point of G. Fici, T. Gagie, J. Kärkkäinen and D. Kempa [7] for providing an algorithm determining, in time $O(n \log(n))$ and space $O(n)$, the palindromic length of a word of length n . In the article [16], A. Shur and M. Rubinchik

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provide another algorithm with the same complexities using a structure storing palindromes. They also conjecture that the palindromic length can be computed in time $O(n)$.

Let us recall that, for any nonempty word v , the infinite periodic word with *period* v ($v \neq \varepsilon$) is denoted by v^ω . Hence any ultimately periodic word is in the form uv^ω for some words u and v ($v \neq \varepsilon$). A.E. Frid, S. Puzynina and L.Q. Zamboni conjectured:

Conjecture 1.1. [9] *If the palindromic lengths of factors of an infinite word \mathbf{w} are bounded (we will say that \mathbf{w} has bounded palindromic lengths of factors) then \mathbf{w} is ultimately periodic.*

In Section 2, we consider several variants of this conjecture and show the equivalence between some of them. In particular, we show that the previous conjecture can be restricted to words having infinitely many palindromic prefixes and that, with this restriction, one should expect words to be periodic instead of being ultimately periodic. This is a consequence of an intermediate result (Lemma 2.7) that states that, if a word has bounded palindromic lengths of prefixes, then it has a suffix that has infinitely many palindromic prefixes. Section 3 shows that this is not just a consequence of the fact that words considered in this result have infinitely many palindromic factors. Indeed we provide an example showing that a word containing infinitely many palindromes may not have a suffix that begins with infinitely many palindromes, even if the considered word is uniformly recurrent. In Section 4, we give a characterization of words having infinitely many palindromic prefixes and whose palindromic lengths of prefixes or factors are bounded by 2. As they are all periodic, this proves the conjectures in a special case.

Let us mention two reasons for the difficulty to prove Conjecture 1.1 in the general case. First when the palindromic lengths of prefixes of an infinite word are not bounded, the function that associates with each integer k the length of the smallest prefix having palindromic length k can grow very slowly. For instance, let us consider the Fibonacci infinite word. As it is the fixed point of the morphism φ defined by $\varphi(a) = ab$ and $\varphi(b) = a$, by [9], the palindromic lengths of prefixes are not bounded. Actually, if $m(k)$ denotes the length of the least nonempty prefix of the Fibonacci word with palindromic length k , one can verify that $m(1) = 1$, $m(2) = 2$, $m(3) = 9$, $m(4) = 62$, $m(5) = 297$, $m(6) = 1154$, $m(7) = 5473$. The second source of difficulties lies in the facts that a word may have several minimal palindromic factorizations and that the minimal palindromic factorizations of a word and of its longest proper prefixes are not related. For instance both words $aabaab$ and $aabaaba$ have palindromic length 2. The first word has two corresponding minimal palindromic factorizations: $aabaa.b$ and $aa.baab$. It is the longest proper prefix of $aabaaba$ who admits only one minimal palindromic decomposition: $a.aabaaba$. To cope with the previous difficulty, in Section 5, we introduce greedy palindromic lengths. For instance, the left greedy palindromic length of a word is the number of palindromes in the palindromic decomposition obtained considering iteratively the longest palindromic prefix as first element. We show that if the left (or the right) greedy palindromic lengths of prefixes of an infinite word \mathbf{w} having infinitely many palindromic prefixes are bounded then \mathbf{w} is periodic. As it also implies that palindromic lengths of factors of \mathbf{w} are bounded, this proves the Frid, Puzynina and Zamboni's conjecture in a special case.

We assume readers are familiar with combinatorics on words. We report them to classical surveys, as [14, 15] for instance, for basic definitions as those already used (word, length, palindrome, prefix, suffix, ...). From now on, A denotes an alphabet. If $u = u_1 \cdots u_k$ is a word (with each u_i a letter), we let \tilde{u} denote the mirror image of u that is the word $u_k \cdots u_1$. We also recall that a finite word w is *primitive* if w is not a power of a smaller word. It is well-known that such a primitive word w is not an internal factor of w , that is $ww = xwy$ implies x or y is the empty word.

2 Variants

In this section, we introduce several conjectures related to Conjecture 1.1 and discuss links between them.

Let us first recall some results on palindromic lengths obtained by A.E. Frid, S. Puzynina and L.Q. Zamboni [9]. By a counting argument, they proved that any infinite word with bounded palindromic lengths of prefixes (and not only factors) contains k -powers for arbitrary integers k . Moreover, if the word is not ultimately periodic, each position of the word must be covered by infinitely many runs. Thus the next conjecture could have been formulated in the article [9].

Conjecture 2.1. *If the palindromic lengths of prefixes of an infinite word \mathbf{w} are bounded then \mathbf{w} is ultimately periodic.*

Obviously if Conjecture 2.1 was verified then Conjecture 1.1 would also be verified. Nevertheless we do not know how to prove the converse (except by proving the conjectures). Hence Conjecture 2.1 seems more difficult than Conjecture 1.1.

We let BPLF (resp. BPLP) denote the set of all infinite words whose palindromic lengths of factors (resp. prefixes) are bounded. Thus Conjecture 1.1 (resp. 2.1) could be rewritten: Any word in BPLF (resp. BPLP) is ultimately periodic.

Observe that any element of the sets BPLF and BPLP contains infinitely many palindromes. We let $\mathcal{P}(A^\omega)$ denote the set of all infinite words over the alphabet A having infinitely many palindromic prefixes.

Next result justify our two new variants of Conjecture 1.1.

Lemma 2.2 ([8, Lem. 5.6]). *Any ultimately periodic word belonging to $\mathcal{P}(A^\omega)$ is periodic.*

Conjecture 2.3. *Any infinite word in $BPLF \cap \mathcal{P}(A^\omega)$ is periodic.*

Conjecture 2.4. *Any infinite word in $BPLP \cap \mathcal{P}(A^\omega)$ is periodic.*

As for Conjectures 1.1 and 2.1, if Conjecture 2.4 is verified then Conjecture 2.3 is also verified, and, we do not know any way to prove directly the converse. Conjecture 2.4 seems more difficult than Conjecture 2.3.

Conjectures 2.3 and 2.4 may be easier to tackle as we have more information on the considered word than in Conjectures 1.1 and 2.1.

Proposition 2.5. *Conjecture 1.1 and 2.3 are equivalent.*

Proposition 2.6. *Conjecture 2.1 and 2.4 are equivalent.*

Proposition 2.5 is a corollary of the following result.

Lemma 2.7. *If the palindromic lengths of prefixes (or more generally of factors) of an infinite word \mathbf{w} are bounded, then there exists a suffix \mathbf{w}' of \mathbf{w} having infinitely many palindromic prefixes.*

Proof. Let $I_0 = \{0\}$ and for $k \geq 1$, let

$$I_k = \{i \mid \exists j \in I_{k-1}, \mathbf{w}[j+1..i] \text{ is a palindrome}\}.$$

Note that for $k \geq 1$, I_k is the set of all lengths of prefixes of \mathbf{w} that can be decomposed into k palindromes. If all suffixes of \mathbf{w} have only a finite number of palindromic prefixes, then it can be checked quite directly by induction that, for all $k \geq 0$, I_k is finite. This contradicts the fact that \mathbf{w} belongs to BPLP. Thus there exists a smallest integer $k \geq 1$ such that I_k is infinite and I_{k-1} is finite. So there exists $j \in I_{k-1}$, such that $\mathbf{w}[j+1..\infty]$ has infinitely many palindromic prefixes. \square

Proof of Proposition 2.5. Let us first assume that Conjecture 1.1 holds and let \mathbf{w} be an element in $\text{BPLF} \cap \mathcal{P}(A^\omega)$: \mathbf{w} is ultimately periodic. By Lemma 2.2, \mathbf{w} is periodic: Conjecture 2.3 holds. Observe now that any suffix of an infinite word in BPLF is also in BPLF . Thus by Lemma 2.7, if $\mathbf{w} \in \text{BPLF}$, it has a suffix \mathbf{w}' in $\text{BPLF} \cap \mathcal{P}(A^\omega)$. Hence Conjecture 2.3 implies Conjecture 1.1. \square

Proposition 2.6 is also a consequence of Lemma 2.7. Nevertheless its proof also needs next lemma that states that any suffix of an element of BPLP also belongs to BPLP . For any integer $k \geq 1$, we let $\text{BPLF}(k)$ (resp. $\text{BPLP}(k)$) denote the set of all infinite words such that $|u|_{\text{pal}} \leq k$ for all their factors u (resp. all their prefixes u).

Lemma 2.8. *Let $k \geq 1$ be an integer, let \mathbf{w} be an infinite word and let a be a letter. If $a\mathbf{w} \in \text{BPLP}(k)$, then $\mathbf{w} \in \text{BPLP}(k+1)$*

Proof. Let p be a finite word. If $ap = \pi_1\pi_2 \cdots \pi_k$ with π_1, \dots, π_k palindromes, then $p = \pi_2 \cdots \pi_k$ if $\pi_1 = a$, and $p = \pi'_1 a \pi_2 \cdots \pi_k$ if $\pi_1 = a\pi'_1 a$. Thus if $a\mathbf{w}$ belongs to $\text{BPLP}(k)$, \mathbf{w} belongs to $\text{BPLP}(k+1)$. In particular if $a\mathbf{w} \in \text{BPLP}$ then $\mathbf{w} \in \text{BPLP}$. \square

Proof of Proposition 2.6. First if Conjecture 2.1 is verified, then Lemma 2.2 implies that Conjecture 2.4 is also verified.

Now assume that Conjecture 2.4 is verified. Let \mathbf{w} be in BPLP . By Lemma 2.7, \mathbf{w} has a suffix \mathbf{w}' in $\mathcal{P}(A^\omega)$. Lemma 2.8 implies that \mathbf{w}' belongs to BPLP . Thus as Conjecture 2.4 is verified, \mathbf{w}' is periodic: \mathbf{w} is ultimately periodic. Conjecture 2.1 is verified. \square

It follows immediately from the definition of BPLF , that for $k \geq 1$ and \mathbf{w} an infinite word, if $a\mathbf{w} \in \text{BPLF}(k)$, then $\mathbf{w} \in \text{BPLF}(k)$. Thus one can ask whether Lemma 2.8 can be improved. But, in general, $a\mathbf{w} \in \text{BPLP}(k)$ does not imply $\mathbf{w} \in \text{BPLP}(k)$. For instance $a(\text{abba})^\omega \in \text{BPLP}(2)$ while $(\text{abba})^\omega \in \text{BPLP}(3) \setminus \text{BPLP}(2)$.

3 About the hypothesis of Conjectures 2.3 and 2.4

As any word in BPLF contains infinitely many palindromes, one could ask whether Lemma 2.7 is a special case of a more general result, that is, whether any word containing infinitely many palindromes has a suffix in $\mathcal{P}(A^\omega)$. The word $\prod_{i=1}^\infty ab^i = \text{ababbabbabbba} \cdots$ shows this is not the case. This word is not *recurrent*, that is, it has some factors (at least all its prefixes) that does not occur infinitely often. The following result provides also a negative answer for *uniformly recurrent* words, that is, words whose all factors occur infinitely often with bounded gaps.

Definition 3.1. *Let $(u_n)_{n \geq 0}$ be the sequence of words defined by $u_0 = aa$ and $u_{n+1} = u_n \text{bbabu}_n \widetilde{u_n}$, and let $U = \lim_{n \rightarrow \infty} u_n$.*

Lemma 3.2. *The word U defined above is a binary uniformly recurrent word containing infinitely many palindromes but such that none of its suffixes begins with infinitely many palindromes.*

Proof. One can observe that $|u_n| = 4 \cdot 3^n - 2$ and $U = aa \prod_{n \geq 0} \text{bbabu}_n \widetilde{u_n}$. Thus the sequence $(u_n \widetilde{u_n})_{n \geq 0}$ is an infinite sequence of pairwise different palindromic factors of U . As $(u_n)_{n \geq 0}$ is a sequence of prefixes of U , it follows that the set of factors of U is closed under reversal (any factor v of U occurs in a prefix u_n and so \widetilde{v} occurs in $\widetilde{u_n}$ itself a factor of U).

Let us prove that U is uniformly recurrent. Let n be an integer. There exists an integer k depending on n such that all factors of length n occur in the prefix u_k of U . As the set of factors of U is closed under reversal, all factors of length n also occur in $\widetilde{u_k}$. From the definition of the sequence $(u_i)_{i \geq 0}$, one can inductively prove that each word u_N with $N \geq k$ can be decomposed over the set $\{u_k, u_k b b a b, u_k b a b b, \widetilde{u_k}, \widetilde{u_k} b b a b, \widetilde{u_k} b a b b\}$. Consequently U can also be decomposed over this set and the distance between two occurrences of a same factor of length n is at most $2|u_k| + 4$. The word U is uniformly recurrent.

To prove that any suffix of U has a finite number of palindromic prefixes, let us first observe that

Fact 3.3. *For each prefix p of $u_n \widetilde{u_n}$ ($n \geq 0$), $|p|_{bbab} - |p|_{babb} \geq 0$.*

Proof. We prove the fact by induction on n . As $u_0 = aa$, the fact is immediate for $n = 0$. Let p be a prefix of u_{n+1} . One of the following cases holds:

Case 1: p is a prefix of u_n ,

Case 2: $p = u_n q$ with $q \in \{b, bb, bba\}$,

Case 3: $p = u_n b b a b p'$ with p' a prefix of u_n or

Case 4: $p = u_n b b a b p' q \widetilde{q}$ with p' a prefix of u_n and q such that $u_n = p' q$.

For any word w , let $\Delta(w)$ denote $|w|_{bbab} - |w|_{babb}$. Observe that aa is both a prefix and a suffix of u_n . Thus $\Delta(u_n q) = \Delta(u_n)$ for any q in $\{b, bb, bba\}$. Consequently Fact 3.3 is a consequence of the inductive hypothesis in Cases 1 and 2. In Case 3, one has $\Delta(p) = \Delta(u_n) + \Delta(p') + 1$ and once again the result holds as a direct consequence of the inductive hypothesis. In Case 4, one has $\Delta(q \widetilde{q}) = 0$. Thus $\Delta(p) = \Delta(u_n) + \Delta(p') + \delta$ with $\delta \in \{0, 1, 2\}$ (0 if $babb$ overlaps the end of p' and the beginning of q , 2 if $bbab$ overlaps the end of p' and the beginning of q , 1 otherwise). Once again, the result holds by induction. \square

Let S be a suffix of $U = aa \prod_{n \geq 0} b b a b u_n \widetilde{u_n}$. There exists an integer N and a word s such that $S = s \prod_{n \geq N} b b a b u_n \widetilde{u_n}$. Let $k = \max(0, |s|_{babb} - |s|_{bbab}) = \max(0, \Delta(s))$. A consequence of Fact 3.3 is that, if $p = [\prod_{n=n_1}^{n_2} b b a b u_n \widetilde{u_n}] p'$ with p' a prefix of $b b a b u_{n_1+1} \widetilde{u_{n_1+1}}$, then $\Delta(p) \geq n_2 - n_1 + 1$. Thus for any prefix π of S longer than $s \prod_{n=N}^{N+k} b b a b u_n \widetilde{u_n}$, $|\pi|_{bbab} - |\pi|_{babb} \geq 1$ and so p cannot be a palindrome: S has a finite number of palindromic prefixes. This ends the proof of Lemma 3.2. \square

4 Another question and a study of bound 2

We have already mentioned that, as $\text{BPLF} \subseteq \text{BPLP}$, Conjecture 1.1 implies Conjecture 2.3, and Conjecture 2.1 implies Conjecture 2.4. Conversely we do not know whether $\text{BPLP} \subseteq \text{BPLF}$. Such a result would be useful to show the equivalence between all previous conjectures. In this context, as for all $k \geq 1$, $\text{BPLF}(k) \subseteq \text{BPLP}(k)$, the following question is of interest.

Question 4.1. *Given an integer $k \geq 1$, what is the smallest integer K (if it exists) such that $\text{BPLP}(k) \subseteq \text{BPLF}(K)$?*

As $\text{BPLP}(1) = \text{BPLF}(1) = \{a^\omega \mid a \text{ letter}\}$, this question holds for $k \geq 2$. Proposition 4.2 answers it for $k = 2$ in the restricted context of words of $\mathcal{P}(A^\omega)$ (a restriction authorized after Section 2 and that reduces the number of cases to study). Note that to prove Proposition 4.2,

we characterize elements of $BPLF(2) \cap \mathcal{P}(A^\omega)$ and $BPLP(2) \cap \mathcal{P}(A^\omega)$. Observing that all these words are periodic confirms the conjectures.

Proposition 4.2. $BPLF(2) \cap \mathcal{P}(A^\omega) \subsetneq BPLP(2) \cap \mathcal{P}(A^\omega) \subsetneq BPLF(3) \cap \mathcal{P}(A^\omega)$.

The proof of this proposition is a direct consequence of Lemma 4.3 and Lemma 4.4 below. Indeed:

- from the definitions, we have $BPLF(2) \subseteq BPLP(2)$;
- from Lemmas 4.3 and 4.4, we observe that $(ababa)^\omega \in BPLP(2) \cap \mathcal{P}(A^\omega) \setminus BPLF(2)$;
- all words occurring in Lemma 4.4 belong to $BPLF(3)$ (the exhaustive verification of palindromic lengths of factors is left to readers);
- the word $(abba)^\omega$ belong to $BPLF(3) \cap \mathcal{P}(A^\omega) \setminus BPLP(2)$ (the exhaustive verification of palindromic lengths of factors is left to readers).

Lemma 4.3. *The set $(BPLF(2) \setminus BPLF(1)) \cap \mathcal{P}(A^\omega)$ is the set of all words in the form $a^i(ba^j)^\omega$ with a, b two different letters, and i, j two integers such that $0 \leq i \leq j$ and $j \neq 0$.*

Proof. Let \mathbf{w} be a word in $(BPLF(2) \setminus BPLF(1)) \cap \mathcal{P}(A^\omega)$. As \mathbf{w} begins with infinitely many palindromes, each factor occurs infinitely often. Condition $\mathbf{w} \in BPLF(2)$ implies that \mathbf{w} is a binary word. Indeed, otherwise it would have a factor in the form $ab^i c$ for an integer $i \geq 1$ and some different letters a, b and c .

If \mathbf{w} has no factor in the form aa with a a letter, as \mathbf{w} is written exactly on two letters (otherwise it would belong to $BPLF(1)$), $\mathbf{w} = a^i(ba)^\omega$ with $i \leq 1$ and a, b two different letters. The result holds.

From now on assume that \mathbf{w} has a factor in the form aa with a a letter. Let b be the second letter occurring in \mathbf{w} . The word \mathbf{w} has a factor in the form $ba^j b$ with $j \geq 2$ (we have initially pointed out that all factors, hence factors aa and b , occur infinitely often in \mathbf{w}). Let \mathbf{w}' be a suffix of \mathbf{w} beginning with $ba^j b$. Let p_n be the smallest prefix of \mathbf{w}' that contains exactly n occurrences of the letter b : $p_0 = \varepsilon$, $p_1 = b$, $p_2 = ba^j b$.

Observe that:

1. for all $k \geq 2$, $|ba^j b^k a|_{pal} = 3$;
2. for all $k \geq 1$, $|ba^j ba^k b|_{pal} = 3$ if $j \neq k$.

Using these observations, it follows by induction that $p_n = (ba^j)^{n-1} b$ for all $n \geq 1$. Hence $\mathbf{w}' = (ba^j)^\omega$.

As \mathbf{w} begins with infinitely many palindromes, prefixes of \mathbf{w} are mirror images of some factors of \mathbf{w}' . Thus $\mathbf{w} = a^i(ba^j)^\omega$ for some integer i such that $0 \leq i \leq j$. \square

Next result provides a characterization of words in $BPLP(2) \cap \mathcal{P}(A^\omega)$.

Lemma 4.4. *An infinite word \mathbf{w} beginning with the letter a and having infinitely many palindromic prefixes is in $BPLP(2)$ if and only if it is in one of the following forms (b is a letter different from a):*

1. $\mathbf{w} = a^\omega$;
2. $\mathbf{w} = (a^i ba^j)^\omega$ for some integers $i \geq 1, j \geq 1$;

3. $\mathbf{w} = (a^i b^j)^\omega$ for some integers $i \geq 1, j \geq 1$;
4. $\mathbf{w} = ((ab)^i a)^\omega$ for some integer $i \geq 2$.

A first step for the proof of the lemma is next result.

Lemma 4.5. *Any infinite word in $BPLP(2) \cap \mathcal{P}(A^\omega)$ contains at most two different letters.*

Proof. Assume that \mathbf{w} contains at least three letters. Then it has a prefix in the form pc with p containing exactly two different letters a and b , and with c a letter different from a and b . Inequality $|pc|_{pal} \leq 2$ implies $|p|_{pal} = 1$.

The word \mathbf{w} has a prefix in the form $pc^i d$ with d a letter different from c , and $i \geq 1$ an integer (As $\mathbf{w} \in \mathcal{P}(A^\omega)$, $\mathbf{w} \neq pc^\omega$). Conditions “ $|pc^i d|_{pal} \leq 2$ ” and “ c does not occur in pd ” implies $p = \varepsilon$, $p = d$ or $p = \pi d$ for a palindrome π . In the latter case, the fact that both π and p are palindromes implies that π is a power of d . In all cases, we have a contradiction with the fact that p contains two different letters. \square

In order to prove Lemma 4.4, we consider the possible palindromic prefixes of \mathbf{w} . We let $\text{next}(u)$ denote the set of all palindromes π over $\{a, b\}$ such that:

- u is a proper prefix of π (A word u is a *proper prefix* of a word v if there exists a non empty word p such that $v = pu$),
- all proper palindromic prefixes of π are prefixes of u (possibly u itself), and
- $|p|_{pal} \leq 2$ for all the prefixes p of π .

For instance $\text{next}(a^i) = \{a^{i+1}\} \cup \text{next}(a^i b)$.

Lemma 4.6. *For any integer $i \geq 1$,*

1. $\text{next}(a^i b) = \{a^i b^j a^i \mid j \geq 1\} \cup \{a^i (ba^j)^k ba^i \mid k \geq 1, 1 \leq j < i\}$;
2. $\text{next}(a^i (ba^j)^k ba^i) = \emptyset$ when $1 \leq j < i, k \geq 1$;
3. $\text{next}(a^i (b^j a^i)^k a) = \emptyset$ when $j \geq 2$ and $k \geq 1$;
4. $\text{next}(a^i (b^j a^i)^k b) = \{a^i (b^j a^i)^{k+1}\}$ when $j \geq 1$ and $k \geq 1$;
5. $\text{next}(a^i (ba^i)^k a) = \emptyset$ when $i \geq 2$ and $k \geq 2$;
6. $\text{next}(a^i ba^{i+1}) = \{a^i ba^{i+j} ba^i \mid j \geq 1\}$.
7. $\text{next}(a^i (ba^{i+j})^k ba^i) = \{a^i (ba^{i+j})^{k+1} ba^i\}$ when $j \geq 1, k \geq 1$;
8. $\text{next}(a(ba)^k a) = \{(a(ba)^k)^2\}$ when $k \geq 2$.
9. $\text{next}((a(ba)^k)^\alpha) = \{(a(ba)^k)^{\alpha+1}\}$ when $k \geq 2$ and $\alpha \geq 2$

Proof. 1. Let $E_1 = \{a^i b^j a^i \mid j \geq 1\} \cup \{a^i (ba^j)^k ba^i \mid k \geq 1, 1 \leq j < i\}$. We let readers verify that $E_1 \subseteq \text{next}(a^i b)$. Assume by contradiction that there exists a word u in $\text{next}(a^i b) \setminus E_1$. Let $\text{Pref}(E_1)$ denote the set of all prefixes of words in E_1 . The word u must have a prefix p in $(\text{Pref}(E_1) \setminus E_1) \setminus \{a, b\} \setminus \text{Pref}(E_1)$. In other words $p = \pi a$ or $p = \pi b$ with $p \notin \text{Pref}(E_1)$, $\pi \in \text{Pref}(E_1)$ and $\pi \notin E_1$. Observe that $\text{Pref}(E_1) = \{a^\ell \mid \ell \leq i\} \cup \{a^i b^j \mid j \geq 0\} \cup \{a^i b^j a^k \mid j \geq 1, 0 \leq k \leq i\} \cup \{a^i (ba^j)^k ba^\ell \mid k \geq 0, 0 \leq \ell \leq i, 1 \leq j < i\}$. As $a^i b$ is a prefix of u and $\pi \notin E_1$, we get $p = a^i b^j a^k b$ with $j \geq 2$ and $1 \leq k < i$, or, $p = a^i (ba^j)^k ba^\ell b$ with $1 \leq j < i, k \geq 1, 0 \leq \ell < i$ and $\ell \neq j$. In these two cases, $|p|_{pal} = 3$. Hence $\text{next}(a^i b) = E_1$.

2. Observe that any palindrome having $a^i(ba^j)^kba^i$ as a proper prefix must have a prefix in one of the two following forms: $p = a^i(ba^j)^kba^{i+\ell}b$ with $\ell \geq 1$; $p = a^i(ba^j)^kba^ib^ma$ with $m \geq 1$. As $1 \leq j < i$ and $k \geq 1$, in both cases $|p|_{pal} = 3$. Thus $\text{next}(a^i(ba^j)^kba^i) = \emptyset$.
3. Any element of $\text{next}(a^i(b^ja^i)^ka)$ must have a prefix $p = a^i(b^ja^i)^ka^\ell b$ with $\ell \geq 1$. Once again, as $j \geq 2$ and $k \geq 1$, $|p|_{pal} = 3$. Hence $\text{next}(a^i(b^ja^i)^ka) = \emptyset$.
4. We let readers verify that $a^i(b^ja^i)^{k+1} \in \text{next}(a^i(b^ja^i)^kb)$. Assume there exists a word u in $\text{next}(a^i(b^ja^i)^kb) \setminus \{a^i(b^ja^i)^{k+1}\}$. It has a prefix p in the form $a^i(b^ja^i)^kb^\ell a$ with $\ell \neq j$ and $\ell \geq 1$ or in the form $a^i(b^ja^i)^kb^ja^\ell b$ with $1 \leq \ell < i$. Thus, as $j \geq 1$ and $k \geq 1$, we have $|p|_{pal} = 3$ except if $j = 1$ for the second form. But then u has pa or pb as a prefix with $p = a^i(ba^i)^kba^\ell b$. As $|pa|_{pal} = |pb|_{pal} = 3$, we get a contradiction. Hence $\text{next}(a^i(b^ja^i)^kb) = \{a^i(b^ja^i)^{k+1}\}$.
5. Let $i \geq 2$, $k \geq 2$ and let u be an element of $\text{next}(a^i(ba^i)^ka)$. For some $\ell \geq 1$ and $m \geq 1$, the word $p_1 := a^i(ba^i)^{k-1}ba^{i+\ell}b^ma$ is a prefix of u . Observe that $|p_1|_{pal} \leq 2$ only if $\ell = 1$. Then u has one of the following two words as a prefix: $p_2 := a^i(ba^i)^{k-1}ba^iab^maa$ or $p_3 := a^i(ba^i)^{k-1}ba^iab^mab$. But, as $i \geq 2$ and $k \geq 2$, $|p_2|_{pal} = 3 = |p_3|_{pal}$: a contradiction with $u \in \text{BPLP}(2)$. Thus $\text{next}(a^i(ba^i)^ka) = \emptyset$.
6. We let readers verify that $\{a^iba^{i+j}ba^i \mid j \geq 1\} \subseteq \text{next}(a^iba^{i+1})$. Now assume that u is an element of $\text{next}(a^iba^{i+1})$. It has a prefix $p_1 := a^iba^{i+j}b^\ell a$ for some $j \geq 1$, $\ell \geq 1$. As $|a^iba^{i+j}bb|_{pal} = 3$, we have $\ell = 1$. As $|a^iba^{i+j}ba^mb|_{pal} = 3$ when $m < i$, we deduce that $u = a^iba^{i+j}ba^i$. Hence $\text{next}(a^iba^{i+1}) = \{a^iba^{i+j}ba^i \mid j \geq 1\}$.
7. Let $i \geq 1$, $j \geq 1$, $k \geq 1$. Observe first that $a^i(ba^{i+j})^{k+1}ba^i \in \text{next}(a^i(ba^{i+j})^kba^i)$. Assume by contradiction that there exists an element u in the set $\text{next}(a^i(ba^{i+j})^kba^i) \setminus \{a^i(ba^{i+j})^{k+1}ba^i\}$. It has a prefix in the form $p_1 = a^i(ba^{i+j})^kba^ia^\ell b$ with $\ell \neq j$ or in the form $p_2 = a^i(ba^{i+j})^kba^{i+j}b^n$ with $n \geq 2$. or in the form $p_3 = a^i(ba^{i+j})^kba^{i+j}ba^mb$ with $0 \leq m < i$. Observe that $|p_2|_{pal} = 3$, $|p_3|_{pal} = 3$ and, $|p_1|_{pal} \leq 2$ only if $\ell = 0$. In this latter case, u has $p_1 = a^i(ba^{i+j})^kba^ib$ as a prefix. Consequently for an integer $n \geq 1$, u has also a prefix $p_4 = a^i(ba^{i+j})^kba^ib^n a$. We have $|p_4|_{pal} \leq 2$ only if $n = 1$ and $j = 1$: $p_4 = a^i(ba^{i+1})^kba^iba$. Observe that $|p_4a|_{pal} = |a^i(ba^{i+1})^kba^ibaa|_{pal} = 3$. Thus for an integer $\alpha \geq 1$, $p_5 = a^i(ba^{i+1})^kba^ibab^\alpha a$ is a prefix of u . Now $|p_5|_{pal} \leq 2$ only if $\alpha = 1$ and $i = 1$: $p_5 = (aba)^ka(ba)^3$. Observe that for any $\gamma \geq 3$, $|(aba)^ka(ba)^\gamma|_{pal} = |a(baa)^k(ba)^\gamma b|_{pal} = 2$ and $|a(baa)^k(ba)^\gamma bb|_{pal} = |a(baa)^k(ba)^\gamma a|_{pal} = 3$. Thus u must have all words $(aba)^ka(ba)^\gamma$ as prefixes (for any $\gamma \geq 3$). We get a contradiction with the finiteness of u . Hence $\text{next}(a^i(ba^{i+j})^kba^i) = \{a^i(ba^{i+j})^{k+1}ba^i\}$.
8. Let $k \geq 2$: $(a(ba)^k)^2 \in \text{next}(a(ba)^ka)$. Let u be an element of the set $\text{next}(a(ba)^ka) \setminus \{(a(ba)^k)^2\}$. It has a prefix $p_1 := a(ba)^k(ab)^ib$ with $0 < i \leq k$ or $p_2 = a(ba)^k(ab)^iaa$ with $0 \leq i < k$. Observe that $|p_1|_{pal} = 3$. Also $|p_2|_{pal} = 3$ except when $i = 0$. It follows that u has a prefix $p_3 = a(ba)^ka^\alpha b^\beta a$ with $\alpha \geq 2$, $\beta \geq 1$. This is not possible as $|p_3|_{pal} = 3$. Hence $\text{next}(a(ba)^ka) = \{(a(ba)^k)^2\}$.
9. Let $k \geq 2$ and $\alpha \geq 2$. First $(a(ba)^k)^{\alpha+1} \in \text{next}((a(ba)^k)^\alpha)$. Let u be an element of $\text{next}((a(ba)^k)^\alpha) \setminus \{(a(ba)^k)^{\alpha+1}\}$.
This word u has a prefix in the form $p_1 = (a(ba)^k)^\alpha(ab)^ib$ with $0 \leq i \leq k$ or in the form $p_2 = (a(ba)^k)^\alpha(ab)^iaa$ with $0 \leq i < k$. We have $|p_1|_{pal} \leq 2$ or $|p_2|_{pal} \leq 2$ only if $i = 0$. In this case u has a prefix $p_3 = (a(ba)^k)^\alpha b^\beta a$ with $\beta \geq 1$ or

$p_4 = (a(ba)^k)^\alpha a^\beta b$ with $\beta \geq 2$. Observe that $|p_4|_{pal} = 3$ and, $|p_3|_{pal} \leq 2$ only if $\beta = 1$. Thus u has $p_3 = (a(ba)^k)^{\alpha-1} a(ba)^{k+1}$ as a prefix. Let $\gamma \geq 1$. Observe that $|(a(ba)^k)^{\alpha-1} a(ba)^{k+\gamma}|_{pal} = 2$ and $|(a(ba)^k)^{\alpha-1} a(ba)^k (ba)^\gamma b|_{pal} = 2$ while $|(a(ba)^k)^{\alpha-1} a(ba)^{k+\gamma} a|_{pal} = 3$ and $|(a(ba)^k)^{\alpha-1} a(ba)^k (ba)^\gamma b b|_{pal} = 3$. We get a contradiction with the finiteness of u . Hence $\text{next}((a(ba)^k)^\alpha) = \{(a(ba)^k)^{\alpha+1}\}$. \square

Proof of Lemma 4.4. The proof of the *if part* just needs to verify that each prefix of a word in one of the four forms can be decomposed in one or two palindromes. This is done straightforwardly as these prefixes are words in one of the following forms:

- $a^i, i \geq 0$;
- $a^i (ba^{i+j})^k b a^i a^\ell, i \geq 1, k \geq 0, j \geq 0, \ell \geq 0$;
- $a^{i-\ell} a^\ell (ba^{i+j})^k b a^\ell, i \geq 1, 0 \leq \ell \leq i, j \geq 0, k \geq 0$;
- $a^{i-\ell} a^\ell (b^j a^i)^k b^j a^\ell, i \geq 1, \ell \leq i, j \geq 1, k \geq 0$;
- $(a^i b^j)^k a^i b^\ell, i \geq 1, j \geq 1, k \geq 0, \ell \leq j$,
- $((ab)^i a)^k (ab)^\ell a, i \geq 0, k \geq 0, \ell \geq 0$;
- $(ab)^{i-\ell} a. (ba)^\ell ((ab)^i a)^k (ab)^\ell, i \geq 1, 0 \leq \ell \leq i, k \geq 0$.

We now prove the only if part. Let \mathbf{w} be an infinite word in $\text{BPLP}(2) \cap \mathcal{P}(A^\omega)$ beginning with the letter a . By Lemma 4.5, \mathbf{w} contains at most two different letters. If \mathbf{w} is written only on one letter, $\mathbf{w} = a^\omega$. Assume \mathbf{w} is written on two letters. Let b be the second letter. For some integer $i \geq 1$, \mathbf{w} begins with $a^i b$. By Items 1 and 2 of Lemma 4.6, for some integer $j \geq 1$, \mathbf{w} begins with $a^i b^j a^i$. Assume $\mathbf{w} \neq (a^i b^j)^\omega$. By Item 4 of Lemma 4.6, \mathbf{w} begins with $a^i (b^j a^i)^k a$ for some $k \geq 1$. By Item 3 of Lemma 4.6, $j = 1$. Assume $k = 1$. By Item 6 of Lemma 4.6 there exists an integer $j' \geq 1$ such that \mathbf{w} begins with $a^i b^{i+j'} b a^i$. Thus by Item 7 of Lemma 4.6, $\mathbf{w} = (a^i b a^{j'})^\omega$. Assume from now on that \mathbf{w} begins with $a^i (b a^i)^k a$ with $k \geq 2$. By Item 5 of Lemma 4.6, $i = 1$. By Items 8 and 9 of Lemma 4.6, $\mathbf{w} = (a(ba)^k)^\omega$. \square

Maybe the main interest of the previous proof is the idea of studying the links between successive palindromic prefixes of the considered infinite words. Nevertheless determining $\text{BPLF}(3)$ or $\text{BPLP}(3)$ seems much more difficult due to a combinatorial explosion even when restricted to binary words. Contrarily to words in $\text{BPLF}(2)$, words in $\text{BPLP}(3)$ (and so in $\text{BPLF}(3)$) contains ternary words. For instance the word $(abac)^\omega$, with a, b and c three different letters, belong to $\text{BPLP}(3)$.

5 Greedy palindromic lengths

We now introduce greedy palindromic lengths. In this section we provide generalities on these notions. In the next sections, we will show that any infinite word with left or right bounded greedy palindromic lengths is ultimately periodic. This proves Conjecture 1.1 in a special case and reinforces all conjectures studied in this paper.

5.1 Definition and examples

Let us define inductively the *left greedy palindromic length* of a word w by: $|\varepsilon|_{LGPal} = 0$, and, $|w|_{LGPal} = 1 + |u|_{LGPal}$ when $w \neq \varepsilon$ and $w = \pi u$ with π the longest palindromic prefix of w . For instance, $|abaa|_{LGPal} = 2$ and $|abaab|_{LGPal} = 3$. Similarly, we define the *right greedy palindromic length* $|w|_{RGPal}$ considering at each step the longest palindromic suffix: $|abaa|_{RGPal} = 3$ and $|abaab|_{RGPal} = 2$. An important difference between greedy palindromic lengths and the palindromic length is that the definition implies a unique decomposition. For a finite word w , we say that (π_1, \dots, π_k) is the left greedy palindromic decomposition of w , if $w = \pi_1 \cdots \pi_k$ and, moreover, for all i , $1 \leq i \leq k$, π_i is the longest palindromic prefix of $\pi_i \cdots \pi_k$. The right greedy palindromic decomposition of a word can be defined similarly.

In order to provide examples and following the idea of Section 4, let us consider an infinite word such that, for all prefixes p , $|p|_{LGPal} \leq 2$. For some letters a and b and integers $n \geq 0$, $k \geq 1$, this word is in the form $a^n b^\omega$, $(ab^k)^\omega$, $(ab^k)^n a^\omega$ or $(ab^k)^n ab^\omega$ (only words in the form b^ω or $(ab^k)^\omega$ belong to $\mathcal{P}(A^\omega)$). Let us prove this claim. First all these forms are suitable as prefixes of such words are, for some integer $k \geq 1$, $n \geq 1$, $i \geq 0$, in the form a^n , $a^n b^k$, $(ab^k)^n a a^i$ or $(ab^k)^n a b^i$. Moreover we have: $|a^n|_{LGPal} = 1$; $|a^n b^k|_{LGPal} = 2$; $|(ab^k)^n a|_{LGPal} = 1$; $|(ab^k)^n a a^i|_{LGPal} = 2$ if $i \geq 1$; $|(ab^k)^n a b^i|_{LGPal} = 2$ if $i \geq 1$. No other word is suitable. Indeed if a word is not in these forms, for some integers $k \geq 1$, $n \geq 1$, it has a prefix in the form $a^i b^k a$ with $i \geq 2$, $(ab^k)^n a^i b$ with $i \geq 2$ or $(ab^k)^n a b^i a$ with $i \neq k$. Moreover $|a^i b^k a|_{LGPal} = 3$ if $i \geq 2$; $|(ab^k)^n a^i b|_{LGPal} = 3$ if $i \geq 2$; $|(ab^k)^n a b^i a|_{LGPal} = 3$ if $i \neq k$.

Now let us consider an infinite word such that, for all prefixes p , $|p|_{RGPal} \leq 2$. For some integers $n \geq 1$ and $k \geq 1$, this word is in the form a^ω , $a^n b^\omega$, $a^n (ba^\ell)^\omega$ with $\ell < n$, $(a^n b)^\omega$ or $(a^n b)^k b^\omega$ (only a^ω and $(a^n b)^\omega$ belong to $\mathcal{P}(A^\omega)$). Let us prove this claim. First all these forms are suitable. Indeed prefixes of such words are, for some integers $n \geq 0$, $k \geq 1$, $\ell \geq 0$, in one of the following forms: a^n ; $a^n b^k$; $a^n (ba^\ell)^i b a^j$ with $\ell < n$, $i \geq 0$, $0 \leq j \leq \ell$; $(a^n b)^k a^\ell$ with $\ell \leq n$; $(a^n b)^k b^\ell$. We have for $n \geq 1$ and $k \geq 1$: $|a^n|_{RGPal} = 1$; $|a^n b^k|_{RGPal} = 2$; $|a^n (ba^\ell)^i b a^j|_{RGPal} = 2$ when $\ell < n$, $i \geq 0$, $0 \leq j \leq \ell$; $|(a^n b)^k a^\ell|_{RGPal} = 1$ if $\ell = n$, 2 if $\ell < n$; $|(a^n b)^k b^\ell|_{RGPal} = 2$. No other word is suitable. Indeed if a word is not in this form, for some integers $k \geq 2$, $n \geq 1$, $\ell \geq 1$ it has a prefix in one of the following forms: $a^n b^k a^\ell$ with $\ell \geq n+1$; $a^n b^k a^\ell b$ with $k \geq 2$ and $\ell \neq 0$; $a^n (ba^\ell)^i b a^j$ with $j \geq n+1$ and $\ell \leq n$; $a^n (ba^\ell)^i b a^j b$ with $i \geq 1$, $j \neq \ell$, $j > 0$ and $\ell \neq n$; $a^n (ba^\ell)^i b a^j b a$ with $i \geq 1$ and $j < n$; $a^n (ba^\ell)^i b a^j b b$ with $0 < j < n$. Observe that the right greedy palindromic length is 3 for all these words.

We have already mentioned that the palindromic length of a finite word of length n can be computed in time $O(n \log(n))$ [7, 16]. We claim that greedy palindromic lengths of such a word can be computed in time $O(n)$. As the left greedy palindromic length of a finite word w is the right greedy palindromic length of the mirror word \tilde{w} , we just have to explain this for the right greedy palindromic length. In the article [11], it was explained how to compute in linear time, for a word w , an array LPS that stores for each i , $1 \leq i \leq |w|$, the length of the longest palindromic suffix ending at position i :

$$LPS[i] = \max\{\ell \mid w[i - \ell + 1..i] \text{ is a palindrome}\}$$

Applying next algorithm after computing this array LPS allows to compute the right greedy palindromic length of w (the result is store in variable RGPal). The whole computation takes an $O(|w|)$ time.

```
RGPal ← 0
i ← |w|
```

while $i > 0$ do:
 $\text{RGPAL} \leftarrow \text{RGPAL} + 1$
 $i \leftarrow i - \text{LPS}[i]$

5.2 Links with palindromic length

The following relation follows directly the definitions of greedy palindromic lengths.

Property 5.1. *For any word u , $|u|_{\text{pal}} \leq \min(|u|_{\text{LGPal}}, |u|_{\text{RGPal}})$.*

Next example, provided to us by P. Ochem, shows that the value $\min(|u|_{\text{LGPal}}, |u|_{\text{RGPal}}) - |u|_{\text{pal}}$ can be arbitrarily large. Let m_n be the n^{th} term of the Multibonacci sequence of words defined over the set of integers viewed as letters by: $m_1 = 1$, $m_{n+1} = m_n(n)m_n$ ($m_2 = 121$, $m_3 = 1213121$, ...). All words m_n end with the letter 1 and readers can verify that $|m_n 1^{-1}|_{\text{pal}} = 2$ while $|m_n 1^{-1}|_{\text{LGPal}} = 2n - 2 = |1^{-1}m_n|_{\text{RGPal}}$. Now let a and b be two symbols that are not integers. For $n \geq 1$, let $M_n = 1^{-1}m_n a b m_n 1^{-1}$. We have $|M_n|_{\text{LGPal}} = 2 + |1^{-1}m_n|_{\text{LGPal}} + |m_n 1^{-1}|_{\text{LGPal}} = 2n + 2 = 2 + |1^{-1}m_n|_{\text{RGPal}} + |m_n 1^{-1}|_{\text{RGPal}} = |M_n|_{\text{RGPal}}$. Moreover $|M_n|_{\text{pal}} = 6$.

A similar example over a binary alphabet can be found in the online proceedings of Conference “Journées Montoises d’Informatique Théorique 2014”. It is also possible to recode previous example applying a morphism on word m_n (as for instance, morphism f_n defined by $f_n(i) = a^{n+1-i}b^{2i}a^{n+1-i}$).

As a consequence of Property 5.1, if for an infinite word \mathbf{w} there exists an integer K such that $|p|_{\text{LGPal}} \leq K$ for any prefix p of \mathbf{w} , then also $|p|_{\text{pal}} \leq K$. We have the following stronger property (let recall that the existence of a similar property when considering the palindromic length instead of the left greedy palindromic length is an open problem):

Property 5.2. *If for an infinite word \mathbf{w} there exists an integer K such that $|p|_{\text{LGPal}} \leq K$ for any prefix p of \mathbf{w} , then also $|u|_{\text{pal}} \leq 2K$ for any factor u of \mathbf{w} .*

Proof. Let u be a factor of \mathbf{w} and let p be such that pu is a prefix of \mathbf{w} . Let π_1, \dots, π_k be the palindromes such that (π_1, \dots, π_k) is the left greedy palindromic decomposition of pu : $k = |pu|_{\text{LGPal}} \leq K$. Let x, y be the words and i be the integer such that $y \neq \varepsilon$, $\pi_i = xy$ and $u = y\pi_{i+1} \dots \pi_k$. The word \tilde{y} is a prefix of the palindrome π_i . Hence $\pi_1 \dots \pi_{i-1}\tilde{y}$ is a prefix of \mathbf{w} . Let π'_1, \dots, π'_ℓ be the palindromes such that $(\pi'_1, \dots, \pi'_\ell)$ is the left greedy palindromic decomposition of $\pi_1 \dots \pi_{i-1}\tilde{y}$: $\ell = |\pi_1 \dots \pi_{i-1}\tilde{y}|_{\text{LGPal}} \leq K$. Palindromes π_j and π'_j are the longest palindromic prefixes of respectively $\pi_j \dots \pi_k$ and $\pi'_j \dots \pi'_\ell$. It follows that $i - 1 < \ell$, $\pi_1 = \pi'_1, \dots, \pi_{i-1} = \pi'_{i-1}$ and $\tilde{y} = \pi'_i \dots \pi'_\ell$. Hence $u = \pi'_i \dots \pi'_\ell \pi_{i+1} \dots \pi_k$ is the product of at most $2K$ palindromes. \square

An interesting question is whether Property 5.2 is still true when considering right greedy palindromic length instead of left greedy palindromic length. This is an open problem.

6 Right greedy palindromic length

In this section, we prove Conjecture 1.1 in the special case where the palindromic length is replaced with the right greedy palindromic length. For an infinite word \mathbf{w} , let $\text{MaxRGPalPref}(\mathbf{w})$ denote the supremum of the set $\{|p|_{\text{RGPal}} \mid p \text{ prefix of } \mathbf{w}\}$.

Theorem 6.1. *Let \mathbf{w} be a non ultimately periodic infinite word. Then $\text{MaxRGPalPref}(\mathbf{w})$ is infinite, that is, there exist prefixes of \mathbf{w} with arbitrarily large right greedy palindromic lengths.*

Proof. Assume first that the word \mathbf{w} has no suffix in $\mathcal{P}(A^\omega)$. By Lemma 2.7, the palindromic lengths of prefixes of \mathbf{w} are unbounded. Property 5.1 implies that the right greedy palindromic lengths of prefixes of \mathbf{w} are unbounded.

Assume from now on that $\mathbf{w} = u\mathbf{w}'$ with u some finite word and $\mathbf{w}' \in \mathcal{P}(A^\omega)$.

Let (π_1, \dots, π_n) be the right greedy palindromic decomposition of the word u . Let recall that this means that $n = |u|_{RGP\text{al}}$, $u = \pi_1 \cdots \pi_n$ and, for i , $1 \leq i \leq n$, π_i is the longest palindromic suffix of $\pi_1 \cdots \pi_i$.

More generally, assume we have found a prefix p of \mathbf{w} with $|p| \geq |u|$ such that its right greedy palindromic decomposition is in the form (π_1, \dots, π_k) with $k = |p|_{RGP\text{al}} \geq n$. Here we mean that this decomposition begins with (π_1, \dots, π_n) , the right greedy palindromic decomposition of u . In three steps, we are going to show how to find a palindrome π_{k+1} such that $p\pi_{k+1}$ is a prefix of \mathbf{w} and $(\pi_1, \dots, \pi_{k+1})$ is the right greedy palindromic decomposition of $p\pi_{k+1}$. Thus $|p\pi_{k+1}|_{RGP\text{al}} = k + 1$. The proof of the theorem follows this construction by induction.

Step 1: *Marking a first occurrence of a new factor v .*

Let x be the word of length $|\pi_1 \cdots \pi_k| + 1$ such that $\pi_1 \cdots \pi_k x$ is a prefix of \mathbf{w} . This word $\pi_1 \cdots \pi_k x$ contains at most $|x|$ different factors of length $|x|$. As \mathbf{w} is not ultimately periodic, by the celebrated Morse-Hedlund theorem (see, e.g., [1, Th. 10.2.6]), the word \mathbf{w} has at least $|x| + 1$ different factors of length $|x|$. Hence there exists a word v of length $|x|$ that does not occur in $\pi_1 \cdots \pi_k x$.

Step 2: *Constructing π_{k+1} .*

Let y be the smallest word such that $\pi_1 \cdots \pi_k y v$ is a prefix of \mathbf{w} (remarks: the word v may overlap the word x ; as $x \neq v$, $|y| \geq 1$). Let also \mathbf{w}'' denote the word such that $\mathbf{w} = \pi_1 \cdots \pi_k y v \mathbf{w}''$. As $|\pi_1 \cdots \pi_k| \geq |u|$, $y v \mathbf{w}''$ is a suffix of \mathbf{w}' . As $\mathbf{w}' \in \mathcal{P}(A^\omega)$, $y v \mathbf{w}''$ also belongs to $\mathcal{P}(A^\omega)$. Hence there exists a palindrome π_{k+1} which is a prefix of $y v \mathbf{w}''$ of length at least $|y v|$.

Step 3: *The word π_{k+1} is the longest palindromic suffix of $\pi_1 \cdots \pi_k \pi_{k+1}$.*

Let π be the longest palindromic suffix of $\pi_1 \cdots \pi_{k+1}$ ($|\pi| \geq |\pi_{k+1}|$) and let z be the word such that $\pi_1 \cdots \pi_{k+1} = z\pi$. As π_{k+1} ends with $\tilde{v}\tilde{y}$, the palindrome π also ends with $\tilde{v}\tilde{y}$. Thus the word zyv is a prefix of $z\pi$ and so of \mathbf{w} . By construction, $\pi_1 \cdots \pi_k y$ is the smallest word such that $\pi_1 \cdots \pi_k y v$ is a prefix of \mathbf{w} . Hence, $|z| \geq |\pi_1 \cdots \pi_k|$ and so $|\pi| \leq |\pi_{k+1}|$: $\pi = \pi_{k+1}$. \square

7 Left greedy palindromic length

In this section, we prove next result, that is, Conjecture 1.1 in the special case where the palindromic length is replaced with the left greedy palindromic length. For an infinite word \mathbf{w} , we let $\text{MaxLGP\text{al}Pref}(\mathbf{w})$ denote the supremum of the set $\{|p|_{LGP\text{al}} \mid p \text{ prefix of } \mathbf{w}\}$.

Theorem 7.1. *Let \mathbf{w} be a non ultimately periodic infinite word. Then $\text{MaxLGP\text{al}Pref}(\mathbf{w})$ is infinite, that is, the left greedy palindromic lengths of prefixes of \mathbf{w} are unbounded.*

The proof of this result appears to be much more difficult than the proof of Theorem 6.1, its analogue with the right greedy palindromic length. Section 7.2 explains the main idea of our strategy and the difficulties to apply it. This section also provides intermediary tools. Section 7.3 contains the proof of Theorem 7.1.

Before going further, let us explain that we can reduce the proof of this theorem to words in $\mathcal{P}(A^\omega)$, that is, to words beginning with infinitely many palindromes. Let \mathbf{w} be a non ultimately periodic infinite word. By Property 5.1 and Lemma 2.7, if \mathbf{w} has no suffix in $\mathcal{P}(A^\omega)$, then the palindromic lengths of prefixes of \mathbf{w} are unbounded.

Thus we can assume that \mathbf{w} has a suffix in $\mathcal{P}(A^\omega)$. Next lemma shows that, when proving Theorem 7.1, one can replace \mathbf{w} by one of its suffixes in $\mathcal{P}(A^\omega)$.

Lemma 7.2. *Let \mathbf{w} be an infinite word having a suffix in $\mathcal{P}(A^\omega)$. There exist palindromes π_1, \dots, π_k and an infinite word \mathbf{w}' in $\mathcal{P}(A^\omega)$ such that $\mathbf{w} = \pi_1 \cdots \pi_k \mathbf{w}'$ and, for all $1 \leq i \leq k$, π_i is the longest palindromic prefix of $\pi_i \cdots \pi_k \mathbf{w}'$. Moreover for any prefix p of \mathbf{w}' , $|p|_{LGP al} = |\pi_1 \cdots \pi_k p|_{LGP al} - k$.*

Proof. Observe that a word \mathbf{w} has a (unique) largest palindromic prefix if and only if it has finitely many palindromic prefixes, that is, it does not belong to $\mathcal{P}(A^\omega)$.

When $\mathbf{w} \notin \mathcal{P}(A^\omega)$, the word \mathbf{w} can be decomposed $\mathbf{w} = \pi_1 \mathbf{w}_1$ with π_1 the largest palindromic prefix of \mathbf{w} and \mathbf{w}_1 an infinite word. Moreover for any prefix p of \mathbf{w}_1 , $|p|_{LGP al} = |\pi_1 p|_{LGP al} - 1$.

Iterating this process, one can find palindromes π_1, \dots, π_k and infinite words $\mathbf{w}_1, \dots, \mathbf{w}_k$ such that: $\mathbf{w} = \pi_1 \cdots \pi_i \mathbf{w}_i$ for all i , $1 \leq i \leq k$; $\mathbf{w}_i \notin \mathcal{P}(A^\omega)$, for all i , $1 \leq i \leq k-1$; π_j is the longest palindromic prefix of $\pi_j \cdots \pi_i \mathbf{w}_i$, for all i, j , $1 \leq j \leq i \leq k$; for any prefix p of \mathbf{w}_i , $|p|_{LGP al} = |\pi_1 \cdots \pi_i p|_{LGP al} - i$. The iteration ends when \mathbf{w}_k belongs to $\mathcal{P}(A^\omega)$. This must occur as \mathbf{w} has a suffix in $\mathcal{P}(A^\omega)$ and, as any suffix of a word in $\mathcal{P}(A^\omega)$ belongs to $\mathcal{P}(A^\omega)$. Taking $\mathbf{w}' = \mathbf{w}_k$ ends the proof of the lemma. \square

7.1 Tools

For a word in $\mathcal{P}(A^\omega)$, we will need to consider the *length increasing sequence of palindromic prefixes* of \mathbf{w} , that is the sequence of palindromic prefixes $(\pi_n)_{n \geq 1}$ of \mathbf{w} such that all palindromic prefixes of \mathbf{w} occur in the sequence and $(|\pi_n|)_{n \geq 0}$ is (strictly) increasing. We will use Lemma 7.4 that characterizes periodicity in the context of words in $\mathcal{P}(A^\omega)$. It will be useful in our last section and may be important in a more general context. For the proof, we need next result.

Lemma 7.3 ([5, Th. 4]). *An infinite periodic word w^ω with w primitive contains infinitely many palindromes if and only if w is the product of two palindromes.*

Lemma 7.4. *Let \mathbf{w} be an infinite word in $\mathcal{P}(A^\omega)$. Let $(\pi_i)_{i \geq 0}$ be its sequence of length increasing palindromic prefixes. The sequence $(|\pi_{i+1}| - |\pi_i|)_{i \geq 0}$ is not decreasing. Moreover this sequence is bounded if and only if \mathbf{w} is periodic.*

Proof. Assume by contradiction that there exists an integer j such that $|\pi_{j+2}| - |\pi_{j+1}| < |\pi_{j+1}| - |\pi_j|$. By [9, Lem. 2] it is known that if $u = vs$ for two palindromes u and v and a word s , then $|s|$ is a period of u (here an integer p is a *period* of a finite word w , if for all integers i between 1 and $|w| - p$, $w[i] = w[i + p]$). Let $p = |\pi_{j+2}| - |\pi_{j+1}|$. From the previous recalled result, p is a period of π_{j+2} and so also of π_{j+1} . As $p < |\pi_{j+1}| - |\pi_j| < |\pi_{j+1}|$, we get that the prefix of length $|\pi_{j+1}| - p$ of π_{j+1} is equal to the suffix of same length of this word. Let π be this word. As π_{j+1} is a palindrome, also π is a palindrome. Observe that $|\pi| = |\pi_{j+1}| - p > |\pi_{j+1}| - (|\pi_{j+1}| - |\pi_j|) = |\pi_j|$. Hence $|\pi_j| < |\pi| < |\pi_{j+1}|$: we have found a contradiction with the definition of the sequence $(\pi_i)_{i \geq 0}$.

When the sequence $(|\pi_{i+1}| - |\pi_i|)_{i \geq 0}$ is bounded, it is ultimately periodic. By [9, Lem. 2], its ultimate value p is a period of all palindromic sufficiently large prefixes π_i of \mathbf{w} . Consequently, \mathbf{w} has period p .

Assume \mathbf{w} is periodic. By Lemma 7.3, $\mathbf{w} = (uv)^\omega$ with u and v two palindromes. The sequence $((uv)^j u)_{j \geq 1}$ is a subsequence of $(\pi_i)_{i \geq 0}$. So the sequence $(|\pi_{i+1}| - |\pi_i|)_{i \geq 0}$ is bounded. \square

7.2 Main ideas on the proof of Theorem 7.1

The main idea of our proof of Theorem 7.1 is to find, when possible, situations like this of next lemma. When a word p is a prefix of a word u , we let $p^{-1}u$ denote the suffix s of u such that $u = ps$. When a word s is a prefix of a word u , we let us^{-1} denote the prefix p of u such that $u = ps$.

Lemma 7.5. *Let $\mathbf{w} \in \mathcal{P}(A^\omega)$ and $(\pi_i)_{i \geq 1}$ be the length increasing sequence of palindromic prefixes of \mathbf{w} . Assume there exist infinitely many integers i for which there exists a palindrome p_i which is a factor of π_{i+1} but does not occur in nor does not overlap the prefix π_i of π_{i+1} . The left greedy palindromic lengths of prefixes of \mathbf{w} are not bounded.*

Proof. Let i be an integer and p_i be a palindrome such that p_i is a factor of π_{i+1} that does not occur in nor does not overlap the prefix π_i of π_{i+1} . There exist words x and y such that $\pi_i^{-1}\pi_{i+1} = xp_iy$. The word $\tilde{y}\tilde{p}_i\tilde{x} = \tilde{y}p_i\tilde{x}$ is a prefix of π_{i+1} . By hypothesis, $|\tilde{y}| \geq |\pi_i|$ (otherwise p_i would be a factor of the prefix π_i of π_{i+1} , or would overlap it). Hence $y = z\pi_i$ for a word z ($\tilde{y} = \pi_i\tilde{z}$). Let $\pi = xp_i z$. We have:

$$\pi_{i+1} = \pi_i \pi \pi_i$$

We now prove that π is the longest palindromic prefix of $\pi\pi_i$. Without loss of generality, we assume that there is exactly one occurrence of p_i in $\pi_i xp_i$. Let π' be the longest palindromic prefix of $\pi\pi_i$, and let s be the word such that $\pi's = \pi\pi_i$. As $|\pi'| \geq |\pi|$ and as xp_i is a prefix of π , the word $p_i\tilde{x}$ is a suffix of π' . Consequently $\tilde{s}xp_i$ is a prefix of π_{i+1} . By choice on the occurrence of p_i at the beginning of the paragraph, $|\tilde{s}xp_i| \geq |\pi_i xp_i|$. By definition of π' , we also have $|s| \leq |\pi_i|$. Thus $|s| = |\pi_i|$ and $\pi' = \pi$. That π is the longest palindromic prefix of $\pi\pi_i$ implies that, for any proper prefix p of π_i , we have $|\pi_i \pi p|_{LGP al} = 2 + |p|_{LGP al}$ (let recall that by definition of the sequence $(\pi_i)_{i \geq 1}$, π_i is the longest prefix of π_{i+1}).

As there exist infinitely many couples (i, p_i) , from what precedes, we can construct an infinite sequences $(u_i)_{i \geq 1}$ of prefixes of \mathbf{w} such that, for all $i \geq 1$, there exists an integer $j \geq i$ and a palindrome v_i such that $u_{i+1} = \pi_j v_i u_i$, u_{i+1} is a prefix of $\pi_{j+1} = \pi_j v_i \pi_j$, v_j is the longest palindromic prefix of $v_i \pi_j$ and $|u_{i+1}|_{LGP al} = 2 + |u_i|_{LGP al}$. Left greedy palindromic lengths of prefixes of \mathbf{w} are not bounded. \square

Next result shows an example of use of previous lemma. This result is Theorem 7.1 in the particular case of a word over an infinite alphabet.

Proposition 7.6. *For any infinite word \mathbf{w} over an infinite alphabet, if \mathbf{w} has infinitely many palindromic prefixes, then the set $\{|p|_{LGP al} \mid p \text{ prefix of } \mathbf{w}\}$ is unbounded.*

Proof. Let $(\pi_n)_{n \geq 1}$ be the length increasing sequence of palindromic prefixes of \mathbf{w} . As the alphabet is infinite, there exist infinitely many integers i such that π_{i+1} contains a letter a_i that does not occur in π_i . As letters are palindromes, by Lemma 7.5, $\text{MaxLGP al Pref}(\mathbf{w})$ is infinite. \square

In next example, the situation of Lemma 7.5 does not hold. So, our approach will have to be adapted. This will lead to the study of three different cases at the end of the proof of Theorem 7.1 in next section.

Example 7.7. Let $\pi_0 = aba$; for all $n \geq 1$, $\pi_n = (\pi_{n-1} a^n)^{(+)} = \pi_{n-1} a^{n-1} \pi_{n-1}$ (where $(+)$ denotes the palindromic closure; for a word u , $u^{(+)}$ is the smallest palindrome having u as a prefix):

$$\pi_1 = abaaba$$

$$\pi_2 = abaabaaabaaba$$

$$\pi_3 = abaabaaabaabaaabaabaaba$$

$$\pi_4 = abaabaaabaabaaabaabaaabaabaaabaabaaabaabaabaaba$$

For all $n \geq 1$, the longest palindromic prefix of $\pi_n^{-1}\pi_{n+1}$ is the word a^n . This palindrome overlaps the last occurrence of π_n and its prefix a^{n-1} does not correspond to the first occurrence of this word a^{n-1} in π_{n+1} .

To end with the general ideas of the proof of Theorem 7.1, let us mention that next lemma will allow to consider palindromes in the form $(uv)^k u$ for words p_i when using Lemma 7.5.

Lemma 7.8. *Let \mathbf{w} be a non ultimately periodic word whose left greedy palindromic lengths of prefixes are bounded. There exist palindromes u and v with uv primitive (in particular $uv \neq \varepsilon$) such that, for all $k \geq 0$, $(uv)^k u$ is a factor of \mathbf{w} .*

Proof. Assume first that there exist palindromes u and v with $uv \neq \varepsilon$ such that, for all $k \geq 0$, $(uv)^k u$ is a factor of \mathbf{w} . Let us show that we can assume uv primitive. Assume uv is not primitive and let z be its primitive root (that is the smallest word such that uv is a power of z). There exist words x, y and integers ℓ_1, ℓ_2 such that $z = xy$, $u = (xy)^{\ell_1} x$ and $v = y(xy)^{\ell_2}$. For any $k \geq 0$, $(uv)^k u = (xy)^{k(\ell_1 + \ell_2) + \ell_1} x$. Moreover as x is both a prefix and a suffix of the palindrome u , x itself is a palindrome. Similarly y is a palindrome. Hence replacing u, v with x, y allows to assume that uv is primitive.

We now show, under hypotheses $\text{MaxLGP}(\mathbf{w})$ finite and \mathbf{w} non ultimately periodic, the existence of palindromes u and v with $uv \neq \varepsilon$ such that, for all $k \geq 0$, $(uv)^k u$ is a factor of \mathbf{w} . We act by induction on $\text{MaxLGP}(\mathbf{w})$. First observe that if $\text{MaxLGP}(\mathbf{w}) = 1$, \mathbf{w} does not verify the hypotheses as it is periodic ($\mathbf{w} = a^\omega$ with a letter).

Assume that $\mathbf{w} \notin \mathcal{P}(A^\omega)$. Then $\mathbf{w} = \pi \mathbf{w}'$ with π the longest palindromic prefix of \mathbf{w} . For any prefix p of \mathbf{w}' , we have $|p|_{\text{LGP}} = |\pi p|_{\text{LGP}} - 1$. Hence $\text{MaxLGP}(\mathbf{w}') < \text{MaxLGP}(\mathbf{w})$. If \mathbf{w}' is ultimately periodic, by Lemma 7.3, $\mathbf{w}' = p(uv)^\omega$ with u, v two palindromes such that $uv \neq \varepsilon$ (and p is a word). Thus for all $k \geq 0$, $(uv)^k u$ is a factor of \mathbf{w}' . If \mathbf{w}' is not ultimately periodic, palindromes u and v exist by inductive hypothesis.

From now on, we assume that $\mathbf{w} \in \mathcal{P}(A^\omega)$. Let $(\pi_i)_{i \geq 1}$ be the length increasing sequence of palindromic prefixes of \mathbf{w} . By Lemma 7.4, the sequence $(|\pi_{i+1}| - |\pi_i|)_{i \geq 1}$ is not decreasing. Moreover as \mathbf{w} is not periodic, this sequence is unbounded. By König's lemma (see [15, Prop. 1.2.3]), there exists an infinite word \mathbf{w}_1 such that each of its prefixes p is a proper prefix of $\pi_j^{-1}\pi_{j+1}$ for some $j \geq 1$. By definition of the sequence $(\pi_i)_{i \geq 1}$, for any proper prefix p of $\pi_j^{-1}\pi_{j+1}$, $|p|_{\text{LGP}} = |\pi_j p|_{\text{LGP}} - 1$. Hence $\text{MaxLGP}(\mathbf{w}_1) < \text{MaxLGP}(\mathbf{w})$. If \mathbf{w}_1 is periodic, by Lemma 7.3, $\mathbf{w}_1 = (uv)^\omega$ for some palindromes u, v with $uv \neq \varepsilon$. Thus for all $k \geq 0$, $(uv)^k u$ is a factor of \mathbf{w}_1 . If \mathbf{w}_1 is not periodic, palindromes u and v exist by inductive hypothesis. As factors of \mathbf{w}_1 are factors of \mathbf{w} , the lemma holds. \square

7.3 Proof of Theorem 7.1

Let \mathbf{w} be a non ultimately periodic infinite word. At the beginning of Section 7, we explained that, if \mathbf{w} has no suffix in $\mathcal{P}(A^\omega)$, then $\text{MaxLGP}(\mathbf{w})$ is infinite. Thus we can assume that $\mathbf{w} = u\mathbf{w}'$ for some word u and some infinite word \mathbf{w}' in $\mathcal{P}(A^\omega)$. Lemma 7.2 shows that there exists a suffix \mathbf{w}'' of \mathbf{w}' such that $\text{MaxLGP}(\mathbf{w})$ is infinite if and only if $\text{MaxLGP}(\mathbf{w}'')$ is infinite. Thus from now on we assume that $\mathbf{w} \in \mathcal{P}(A^\omega)$.

We act by contradiction and so we assume that $\text{MaxLGP}(\mathbf{w})$ is finite. Moreover without loss of generality, we assume that if \mathbf{w}' is any infinite word with $\text{MaxLGP}(\mathbf{w}') <$

$\text{MaxLGP}(\mathbf{w})$ then it is ultimately periodic. By Lemma 7.8, there exist palindromes u and v with uv primitive such that, for all $k \geq 0$, $(uv)^k u$ is a factor of \mathbf{w} . Note that factors $(uv)^k u$ are palindromes. Non periodicity of \mathbf{w} also has the following consequence.

Fact 7.9. $\text{Pref}(\mathbf{w}) \cap \text{Fact}((uv)^\omega)$ is finite.

Proof. If $\text{Pref}(\mathbf{w}) \cap \text{Fact}((uv)^\omega)$ is infinite, there exists a suffix s of uv such that $\text{Pref}(\mathbf{w}) \cap \text{Fact}(s(uv)^\omega)$ is infinite. Then $\mathbf{w} = s(uv)^\omega$ and \mathbf{w} is periodic: a contradiction. \square

Notation. To continue the proof of Theorem 7.1, we need to introduce notation. Let L be the length of the greatest prefix of \mathbf{w} that belongs to $\text{Fact}((uv)^\omega)$. Let recall that $\mathbf{w} \in \mathcal{P}(A^\omega)$. We let $(\pi_i)_{i \geq 1}$ denote the sequence of length increasing palindromic prefixes of \mathbf{w} . We now consider first occurrences of words $(uv)^k u$. Let $k \geq 1$, we let p_k denote the smallest prefix of \mathbf{w} such that $p_k(uv)^k u$ is also a prefix of \mathbf{w} . We also let j_k denote the integer such that $|\pi_{j_k}| < |p_k(uv)^k u| \leq |\pi_{j_k+1}|$. In other words, j_k is the greatest integer such that $(uv)^k u$ is not a factor of π_{j_k} . Equivalently $j_k + 1$ is the smallest integer such that $(uv)^k u$ is a factor of π_{j_k+1} .

Fact 7.10. *There exist infinitely many integers i such that $2|\pi_i| < |\pi_{i+1}|$.*

Proof. Consider an integer i such that $i = j_k$ for some integer $j_k \geq 1$. Assume that $2|\pi_i| \geq |\pi_{i+1}|$. (See Fig. 1 for an illustration of the hypotheses and the notation of this proof)

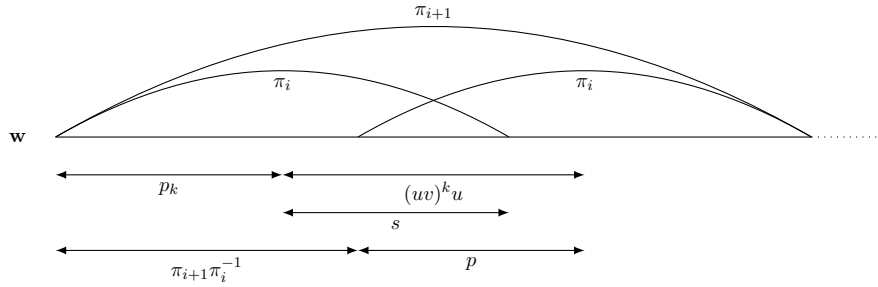


Figure 1: Hypotheses and notation in the proof of Fact 7.10

By choice of i , $|p_k(uv)^k u| \geq |\pi_i| \geq |\pi_{i+1}| - |\pi_i|$. We also have $|p_k| < |\pi_{i+1}| - |\pi_i| \leq |\pi_i|$. Indeed otherwise the word $(uv)^k u$ would be a factor of the suffix π_i of π_{i+1} . This contradicts the definition of $i = j_k$.

Consider the suffix s of π_i such that $\pi_i = p_k s$. As π_i and $p_k(uv)^k u$ are prefixes of π_{i+1} . The word s is a prefix of $(uv)^k u$. As $(uv)^k u$ and π_i are palindromes, \tilde{s} is a prefix of π_i (and so of \mathbf{w}) and a factor of $(uv)^k u$.

Consider now the prefix p of π_i of length $|p_k(uv)^k u| - (|\pi_{i+1}| - |\pi_i|)$: $p_k(uv)^k u = (\pi_{i+1} \pi_i^{-1}) p$. As $|p_k| < |\pi_{i+1}| - |\pi_i| = |\pi_{i+1} \pi_i^{-1}|$, p is a suffix of $(uv)^k u$.

Observe now that $|p| + |\tilde{s}| \geq |(uv)^k u|$. Indeed $|p| + |\tilde{s}| = |p| + |s| = (|p_k(uv)^k u| - |\pi_{i+1}| + |\pi_i|) + (|\pi_i| - |p_k|) = |(uv)^k u| - |\pi_{i+1}| + 2|\pi_i|$, and, $2|\pi_i| \geq |\pi_{i+1}|$.

We have shown that hypothesis $2|\pi_i| \geq |\pi_{i+1}|$ with $i = j_k$ implies that \mathbf{w} has a prefix $(p$ or $\tilde{s})$ of length at least $|(uv)^k u|/2$ which is a factor of $(uv)^k u$. Fact 7.9 shows that this situation can hold only for a finite number of integers k . This ends the proof of Fact 7.10. \square

Intermediate step. Splitting the proof into three cases.

Let us observe now that, in a word π_i (as in any finite word), there occur only finitely many factors in the form $(uv)^k u$. In the following we will always consider integers k such that $(uv)^{k+1}u$ does not occur in π_{j_k} . From Fact 7.10 and its proof, the following set is infinite

$$\mathcal{I} = \{(i, k) \mid i = j_k, 2|\pi_i| < |\pi_{i+1}|, (uv)^{k+1}u \text{ is not a factor of } \pi_{i+1}\}.$$

For any $k \geq 0$ with (j_k, k) in \mathcal{I} , exactly one of the following three cases holds:

Case 1: $|\pi_{j_k}| \leq |p_k|$,

Case 2: $|\pi_{j_k}| > |p_k|$ and $|p_k(uv)^k u| \geq |\pi_{j_k+1}| - |\pi_{j_k}|$,

Case 3: $|\pi_{j_k}| > |p_k|$ and $|p_k(uv)^k u| < |\pi_{j_k+1}| - |\pi_{j_k}|$.

For $n \in \{1, 2, 3\}$, let \mathcal{I}_n be the set of all (i, k) in \mathcal{I} such that case n holds. Sets \mathcal{I}_1 , \mathcal{I}_2 and \mathcal{I}_3 form a partition of \mathcal{I} . So at least one is infinite. The proof of Theorem 7.1 ends with the next three facts, each one proving that one of the sets \mathcal{I}_1 , \mathcal{I}_2 and \mathcal{I}_3 cannot be infinite without contradicting the hypotheses.

Fact 7.11. *Hypothesis “ \mathcal{I}_1 is infinite” is contradictory*

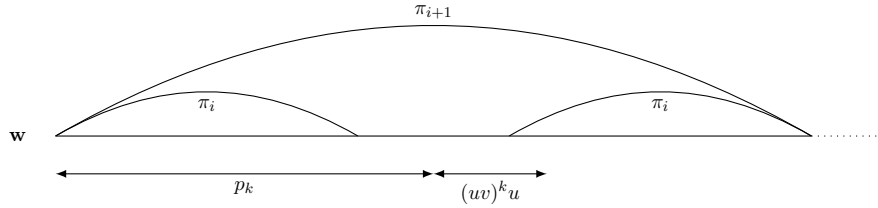


Figure 2: Case 1 with $i = j_k$; elements of \mathcal{I}_1

Proof. Let (i, k) be an element of \mathcal{I}_1 (see Figure 2). We have $|\pi_i| \leq |p_k|$. Let s_k be the word such that $\pi_{i+1} = p_k(uv)^k u s_k$. As π_{i+1} , u and v are palindromes, by definition of p_k , it follows that $|s_k| = |\tilde{s}_k| \geq |p_k|$. As $|\pi_i| \leq |p_k|$, by definition of p_k , the word $(uv)^k u$ occurs in π_{i+1} but does not occur nor overlap the prefix of π_i . Thus if \mathcal{I}_1 is infinite, Lemma 7.5 raises a contradiction with $\text{MaxLGPalPref}(\mathbf{w})$ finite. \square

Fact 7.12. *Hypothesis “ \mathcal{I}_2 is infinite” is contradictory*

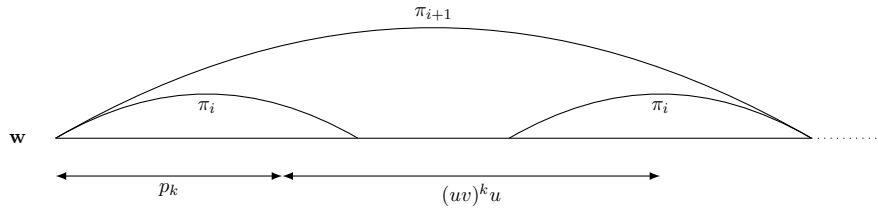


Figure 3: Case 2 with $i = j_k$; elements of \mathcal{I}_2

Proof. Let (i, k) be an element of \mathcal{I}_2 (see Figure 3). Let p be the word such that $\pi_i = p_k p$. As $|p_k(uv)^k u| \geq |\pi_{i+1}| - |\pi_i|$ and $|\pi_{i+1}| > 2|\pi_i|$, $|p_k(uv)^k u| > |\pi_i|$, and so, p is a prefix of $(uv)^k u$. It follows that \tilde{p} belongs to $\text{Pref}(\mathbf{w}) \cap \text{Fact}((uv)^k u)$. By definition of L , $|\tilde{p}| \leq L$. The word p , that depends on k , can take only a finite number of values. Possibly replacing \mathcal{I}_2 by an infinite subset, we assume that this value is the same for all elements of \mathcal{I}_2 . Similarly we can assume that for all $(i, k) \in \mathcal{I}_2$, $k \geq 4L + 6$ and, as $\lim_{k \rightarrow \infty} |\pi_i| = \infty$, $|\pi_i| > 2|p|$. For (i, k) in \mathcal{I}_2 , let π'_i denote the word such that $\pi_i = \tilde{p}\pi'_i p$.

Let s_k be the word such that $\pi_{i+1} = p_k(uv)^k u s_k$. As π_{i+1} , u and v are palindromes, $\pi_{i+1} = \tilde{s}_k(uv)^k u \tilde{p}_k$.

Claim 7.13. $\tilde{s}_k = p_k$

Proof of Claim 7.13. By hypotheses of Case 2, $|p_k(uv)^k u| \geq |\pi_{i+1}| - |\pi_i| = |p_k(uv)^k u s_k| - |\pi_i|$. Thus $|\pi_i| \geq |s_k|$. Words $\tilde{s}_k(uv)^k u$ and $p_k(uv)^k u$ are prefixes of π_{i+1} . Definition of p_k implies that $|s_k| = |\tilde{s}_k| \geq |p_k|$.

Let x be the word such that $\tilde{s}_k = p_k x$. As $|\pi_i| \geq |\tilde{s}_k|$ and $\pi_i = p_k p$, the word x is a prefix of p . Moreover as $\tilde{s}_k(uv)^k u$ and $p_k(uv)^k u$ are prefixes of π_{i+1} , $(uv)^k u$ is a prefix of $x(uv)^k u$. We have restricted the set \mathcal{I}_2 to elements (i, k) such that $k \geq 4L + 6 \geq 1 + |p|$. Thus $k|uv| \geq |uv| + |p| \geq |uv| + |x|$: xuv is a prefix of $(uv)^k$. Let recall that uv is a primitive word. Thus it cannot be an internal factor of $uvuv$. It follows that there exists an integer ℓ such that $x = (uv)^\ell$.

As $\tilde{s}_k(uv)^k u$ is a prefix of π_{i+1} , $(uv)^{\ell+k} u$ is a factor of π_{i+1} . Just before splitting into three cases the proof, we assume that, for $(i, k) \in \mathcal{I}$, $(uv)^{k+1} u$ is not a factor of π_{i+1} . Hence $\ell = 0$ and $\tilde{s}_k = p_k$ \square

Let π be the palindrome such that $\pi_{i+1} = \pi_i \pi \pi_i$. Let recall that $\pi_{i+1} = p_k(uv)^k u \tilde{p}_k$, $\pi_i = \tilde{p}\pi'_i p$, $p_k = \tilde{p}\pi'_i$. Hence $(uv)^k u = p\pi\tilde{p}$. There exist words x, y and integers m, ℓ such that $p = (uv)^m x$ with $|x| < |uv|$, $uv = xy$ ($\tilde{y}\tilde{x} = vu$ as u and v are palindromes), $\pi = y(uv)^\ell u\tilde{y}$. Observe that $(uv)^k u = p\pi\tilde{p} = (uv)^{\ell+2m+2} u$ and so $k = \ell + 2m + 2$. As $k \geq 4L + 6$ and $L \geq m$ (as $|p| \leq L$), we have $\ell \geq 2L + 4$.

Let n be the greatest integer such that $(\tilde{x}\tilde{y})^n$ is a prefix of π_i ($n = 0$, if $\tilde{x}\tilde{y}$ is not a prefix of π_i). Let q denote the word $(\tilde{x}\tilde{y})^n$ ($q = \varepsilon$ if $n = 0$ and $q = \tilde{x}(vu)^{n-1}\tilde{y}$ otherwise). As $q \in \text{Pref}(\mathbf{w}) \cap \text{Fact}((uv)^\omega)$, $|q| \leq L$ and so $n \leq L$. Thus $\ell + n < k$. The word $p_{\ell+n}$ is a prefix of π_i .

As $\ell \geq k - 2L - 2$, $\lim_{k \rightarrow \infty, (i,k) \in \mathcal{I}_2} \ell = +\infty$, and as \mathbf{w} is not ultimately periodic, $\lim_{k \rightarrow \infty, (i,k) \in \mathcal{I}_2} p_{\ell+n} = +\infty$. Possibly removing, once again, a finite number of elements of \mathcal{I}_2 , one can assume $|p_{\ell+n}| \geq |q| + |\tilde{x}\tilde{y}|$.

Claim 7.14. $\pi q = y(uv)^{\ell+n} u\tilde{y}$ is the longest palindromic prefix of $\pi p_{\ell+n}$.

Proof of Claim 7.14. Let π' be the longest palindromic prefix of $\pi p_{\ell+n}$. As $|p_{\ell+n}| \geq |q| + |\tilde{x}\tilde{y}|$, the word π' begins with the palindrome π'' defined by $\pi'' = \pi q = y(uv)^{\ell+n} u\tilde{y}$. As π' is a palindrome, π' ends with π'' . There exists a word z such that $\pi' = zy(uv)^{\ell+n} u\tilde{y}$.

By definition of $p_{\ell+n}$, the word $(uv)^{\ell+n} u$ is not a factor of $p_{\ell+n}$. Thus any occurrence of $(uv)^{\ell+n} u$ in $\pi p_{\ell+n}$ must occur in or overlap the prefix π . Thus $|zy| \leq |\pi|$ and $\pi' = \pi p$ with p both a prefix of $p_{\ell+n}$ and a suffix of $(uv)^{\ell+n} u\tilde{y}$. As p is a factor of $(uv)^\omega$ and a prefix of \mathbf{w} , $|p| \leq L$. As $\ell \geq 2L + 4$, it follows that the prefix $\pi'' = y(uv)^{\ell+n} u\tilde{y}$ of π' overlaps the suffix $(uv)^{\ell+n} u\tilde{y}$ of π' by a factor of length at least $2|uv|$.

As uv is primitive, uv is not an internal factor of $uvuv$. Thus $\pi' = \pi(\tilde{x}\tilde{y})^{n'} = y(uv)^{\ell+n'}\tilde{y}$ for some integer n' . As $\pi(\tilde{x}\tilde{y})^n$ is a prefix of $\pi p_{\ell+n}$, $n' \geq n$. Maximality of n in its definition implies $n' = n$, that is, $\pi' = \pi$ \square

We have already seen that $\lim_{k \rightarrow \infty, (i,k) \in \mathcal{I}_2} \ell = +\infty$. Consequently $\lim_{k \rightarrow \infty, (i,k) \in \mathcal{I}_2} q^{-1}p_{\ell+n} = +\infty$. Let z be a prefix of $q^{-1}\mathbf{w}$. There exist integers i, k, ℓ such that qz is a prefix of $p_{\ell+n}$ itself a prefix of π_i and for a palindrome π , $\pi_{i+1} = \pi_i\pi\pi_i$. Moreover from what precedes, especially from Claim 7.14, $|\pi_i(\pi q)z|_{LGP\text{al}} = 2 + |z|_{LGP\text{al}}$. Hence $\text{MaxLGP\text{al}Pref}(q^{-1}\mathbf{w}) = \text{MaxLGP\text{al}Pref}(\mathbf{w}) - 2 < \text{MaxLGP\text{al}Pref}(\mathbf{w})$. By an initial hypothesis (see the second paragraph of Section 7.3), we deduce that $q^{-1}\mathbf{w}$ is ultimately periodic. So is \mathbf{w} . Hence Hypothesis “ \mathcal{I}_2 is infinite” is contradictory. \square

Fact 7.15. *Hypothesis “ \mathcal{I}_3 is infinite” is contradictory*

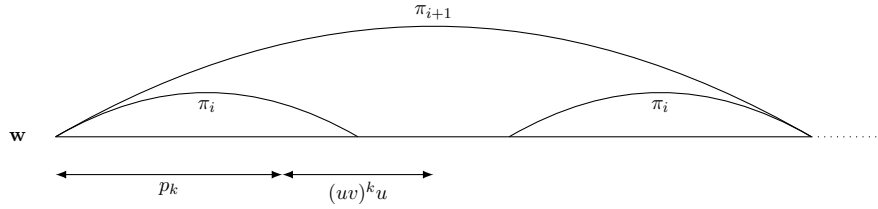


Figure 4: Case 3 with $i = j_k$; elements of \mathcal{I}_3

Proof. Assume by contradiction that \mathcal{I}_3 is infinite. As in the proof of Fact 7.12, possibly replacing \mathcal{I}_3 with an infinite subset, as a consequence of Fact 7.9, we can assume $\ell \geq 2 + L$. Let (i, k) be an element of \mathcal{I}_3 (see Figure 3). As $|\pi_i| > |p_k|$, let p denote the suffix of π_i such that $\pi_i = p_k p$. By definition of p_k and $i = j_k$, $|p_k(uv)^k u| > |\pi_i|$: p is a prefix of $(uv)^k u$. There exist words x, y and integers m, ℓ such that $p = (uv)^m x$, $k = \ell + m + 1$, $uv = xy$, $|x| < |uv|$. Thus $y(uv)^\ell u$ is a prefix of $\pi\pi_i$ where π is the word such that $\pi_{i+1} = \pi_i\pi\pi_i$ (let recall that $2|\pi_i| > |\pi_{i+1}|$).

Let us use the hypothesis $|p(uv)^k u| < |\pi_{i+1}| - |\pi_i| = |\pi_{i+1}\pi_i^{-1}|$. It implies that $y(uv)^\ell u$ is a prefix of π . As π is a palindrome (as $\pi_{i+1} = \pi_i\pi\pi_i$ is a palindrome), $(uv)^\ell u\tilde{y}$ is also a suffix of π . Note that $\ell < k$ and so p_ℓ is a prefix of p_k so of π_i .

Claim 7.16. π is the longest palindromic prefix of πp_ℓ .

Proof of Claim 7.16. Let π' be the longest palindromic prefix of πp_ℓ . As $(uv)^\ell u$ does not occur in p_ℓ , any occurrence of $(uv)^\ell u$ must occur in π or overlap this prefix of πp_ℓ . The word π' ends with $(uv)^\ell u\tilde{y}$. So this occurrence must overlap the suffix $(uv)^\ell u\tilde{y}$ of π . As in the proof of Claim 7.14 in the proof of Fact 7.12, as $\ell \geq 2 + L$, $\pi' = \pi(\tilde{x}\tilde{y})^n$ for some integer n . If $n \geq 1$, then $y(uv)^{\ell+1}u$ is a prefix of π and $p_k(uv)^{k+1}u$ is a prefix of π_{i+1} . This contradicts an earlier hypothesis on $i = j_k$. Thus $y(uv)^{\ell+1}u$ is not a prefix of π : we get $n = 0$. \square

Let us end the proof of Fact 7.15. Observe that $\lim_{k \rightarrow \infty, (i,k) \in \mathcal{I}_3} p_\ell = +\infty$. Thus for any prefix z of \mathbf{w} , we can find an integer i and a palindrome π such that $|\pi_i\pi z|_{LGP\text{al}} = 2 + |z|_{LGP\text{al}}$. It follows that $\text{MaxLGP\text{al}Pref}(\mathbf{w}) = \text{MaxLGP\text{al}Pref}(\mathbf{w}) - 2$. This is impossible \square

8 Conclusion

To summarize this paper, observe that the A. Frid, S. Puzynina and L.Q. Zamboni's conjecture could be reformulated as follows. For an infinite word \mathbf{w} having infinitely many palindromic prefixes, the following assertions are equivalent:

1. \mathbf{w} has bounded palindromic lengths of factors;
2. \mathbf{w} has bounded palindromic lengths of prefixes;
3. \mathbf{w} has bounded left greedy palindromic lengths of factors;
4. \mathbf{w} has bounded left greedy palindromic lengths of prefixes;
5. \mathbf{w} has bounded right greedy palindromic lengths of factors;
6. \mathbf{w} has bounded right greedy palindromic lengths of prefixes;
7. \mathbf{w} is periodic;
8. $\mathbf{w} = (uv)^\omega$ with u and v two palindromes.

Equivalence between assertions 3 to 8 are proved in this paper, and clearly assertion 8 implies assertion 1 which implies assertion 2. That assertion 1 or assertion 2 implies assertion 8 stays an open problem.

To end this paper, let us mention another related problem. We first need notation. For any finite or infinite word w , let $B(w) := \max\{|p|_{pal} \mid p \text{ prefix of } w\}$. For any integer $k \geq 1$, let also $B(k) = \min\{B(\mathbf{w}) \mid \#\text{alph}(\mathbf{w}) = k, \mathbf{w} \text{ infinite}\}$ where $\#$ denotes the cardinality of a set and $\text{alph}(\mathbf{w})$ is the alphabet of \mathbf{w} . The value $B(k)$ is the least value B for which there exists an infinite word \mathbf{w} written using k letters and whose palindromic lengths of prefixes are bounded by B .

Clearly $B(1) = 1$. Let us show that $B(2) = 2$ and $B(3) = 3$. Any infinite binary word begins with a word in the form aa^ib ($i \geq 0$) whose palindromic length is 2: $B(2) \geq 2$. As there exist binary words in $\text{BPLF}(2)$, $B(2) = 2$. Any infinite ternary word begins with a word in the form ucc^ia for some different letters a, b and c with $i \geq 0$, $|u|_a \geq 1$, $|u|_b \geq 1$ and $|u|_c = 0$. Readers can verify that $|uc|_{pal} \geq 3$ if u is not a palindrome and $|ucc^ia|_{pal} \geq 3$ if u is a palindrome. Hence $B(3) \geq 3$. Word $(1213121)^\omega$ belong to $\text{BPLF}(3)$ and so shows that $B(3) = 3$.

Observe also that for any integer $k \geq 1$, $B(k) \leq B(k+1)$. Indeed if \mathbf{w} is a word written using $k+1$ letters, if a is a letter occurring in a , and if $\delta_a(\mathbf{w})$ is the word obtained from \mathbf{w} removing all its occurrences of a , then $B(\delta_a(\mathbf{w})) \leq B(\mathbf{w})$.

Next result shows that $B(2^k) \leq k+1$ and so $B(4) = 3$.

Let inc be the morphism over \mathbb{N}^* defined by $\text{inc}(n) = (n+1)$ for all letters n in \mathbb{N} . Let $u_1 = 1$, and let, for $n \geq 1$, $u_{n+1} = u_n \text{inc}^{2^{n-1}}(u_n) u_n$: $u_2 = 121$, $u_3 = 121343121$, $u_4 = 121343121565787565121343121$, ... It is straightforward that u_n is written using 2^n letters and is of length 3^{n-1} . Let also $v_n = \text{inc}^{2^{n-1}}(u_n)$. Note that all words u_n and v_n are palindromes.

Lemma 8.1. *For $n \geq 1$, $B(u_n) = n$ and $B((u_n v_n)^\omega) = n+1$.*

Proof. The proof acts by induction. The result is clearly true for $n = 1$. Assume $B(u_n) = n$ for some integer $n \geq 1$. It follows that $B(v_n) = n$ also. Let p be a prefix of $(u_n v_n)^\omega$. If p is a prefix of u_n , $|p|_{pal} \leq n$ by hypothesis. If $p = (u_n v_n)^\ell p_1$ with p_1 a prefix of u_n and $\ell \geq 1$, let p_2 be the word such that $u_n = p_2 \tilde{p}_1$: $p = p_2 (\tilde{p}_1 v_n (u_n v_n)^{\ell-1} p_1)$. As $\tilde{p}_1 v_n (u_n v_n)^{\ell-1} p_1$ is a

palindrome, $|p|_{pal} \leq |p_2|_{pal} + 1 \leq n + 1$. Finally, if $p = (u_n v_n)^\ell u_n p_1$ with p_1 a prefix of v_n and $\ell \geq 0$, as $(u_n v_n)^\ell u_n$ is a palindrome, $|p|_{pal} \leq |p_1|_{pal} + 1 \leq n + 1$. Hence $B((u_n v_n)^\omega) \leq n + 1$.

Now let p be a prefix of v_n such that $|p|_{pal} = n$. As u_n is a palindrome, and as the letters in u_n and in p are different, $|u_n p|_{pal} = 1 + |p|_{pal}$. Hence $B((u_n v_n)^\omega) = n + 1$.

Now as $u_n p$ is a prefix of u_{n+1} which itself is a prefix of $(u_n v_n)^\omega$, $B(u_{n+1}) = n + 1$. \square

Now we are able to state our new problem: for $k \geq 1$ and for i such that $2^{k-1} < i \leq 2^k$, is it true that $B(i) = k + 1$?

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